

A GENERALIZED BACKWARDS SCHEME FOR SOLVING NON MONOTONIC STOCHASTIC RECURSIONS

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ABSTRACT. We propose an explicit construction of a stationary solution for a stochastic recursion of the form $X \circ \theta = \varphi(X)$ on a partially-ordered Polish space, when the monotonicity of φ is not assumed. Under certain conditions, we show that an extension of the original probability space exists, on which a solution is well-defined, and construct explicitly this extension. We then provide conditions for the solution to be defined as well on the original space. We finally apply these results to the stability study of two non-monotonic queueing systems.

Keywords: Stochastic recursions, Stationary solutions, Enriched probability space, Ergodic Theory, Queueing Theory.

1. INTRODUCTION

The evolution of a number of dynamical systems depend on punctual, random perturbations which may be assumed time-stationary. In such cases, the state of the system can be described, in discrete time, by a random sequence generated by a recursive, random functional termed *driving mapping* of the recursion:

$$X_{n+1} = \varphi_n(X_n), n \geq 0.$$

In the general framework (of crucial interest in the application), where the sequence $\{\varphi_n\}$ driving the recursion is time-stationary but not necessarily independent, we adopt an ergodic-theoretical approach to formally address the central question of *stability*, *i.e.* of existence of an equilibrium state for the recursion.

It is well-known since the pioneering works of Loynes (see [11] and among others, [4]), that a stationary state exists whenever the random maps φ_n enjoy mild properties, such as (i) monotonicity and continuity, as assumed by Loynes, or (ii) some regenerative property, as in Borovkov's Theory of Renovating Events (see [6]). Notice that the latter framework is also suitable, under certain conditions, for random sequences that are not stochastically recursive, see [7].

However, a lot of (even very simple) models don't verify such assumptions. A classical example is the well-known so-called *Loss queueing system*, addressed in section 5). It is easy to construct cases in which either none, or several stationary states may exist. For this particular model, Neveu [13] and Flipo [8, 9] have shown that the stability problem can be solved at least on a larger probability space. Their constructions, inspired by skew-product methods used to solve ordinary or partial differential equations, lead to an *extension* (also called *enrichment*) of the original probability space on which a stationary solution exists (see as well Lisek [10] for related developments).

More recently, Anantharam and Konstantopoulos [1, 2] show that such extensions exist under mild assumption on the statistics of the recursion, using an approach based on tightness properties. The construction presented in [1, 2], although more general, is less

tractable in that the probability measure on the extension (termed *weak solution*) is identified as a weak limit, and is not explicitly defined.

Following the same directions, we aim to identify the conditions of existence of such extensions, for a more general class of models. We also propose, under such conditions, a constructive scheme of the enriched probability space - see Theorem 1 below. Our framework appears particularly adequate, when coming back to the original problem: it leads to several sufficient conditions of existence of a stationary state on the original probability space (see Proposition 3). Then, Loynes's Theorem and Borovkov and Foss's Theorem of Renovating events turn out to be particular cases of our result (see subsections 4.3 and 4.4). As a matter of fact, the three approaches all rely on the same time reversal technique (usually termed *Backwards scheme*). We therefore term our construction *Generalized backwards scheme*.

The outline of this paper is the following. After introducing our main notation and assumptions in section 2, we give in section 3 a sufficient condition of existence of an extension solving the recursion, based on the tightness argument of Anantharam and Konstantopoulos (*ibid*). The main result of this work is presented in section 4: we construct explicitly the extension, and deduce several conditions for solving the original stability problem. We conclude with two cases study: in section 5 we handle in this framework the stability problem of the Loss queueing system. Finally, in Section 6 we address the same problem for a generalization of this model: the Queue with impatient customers.

2. PRELIMINARY

Let E be a Polish space that is endowed with a partial ordering \preceq . For all $x, y \in E$ such that $x \preceq y$, we denote

$$[[x, y]] := \{z \in E; x \preceq z \preceq y\}.$$

We assume that E admits a \preceq -minimal point denoted 0_E , and is *Lattice-ordered*: any \preceq -increasing sequence converges (possibly to some element of the adherence of E). Any subset $A \subset E$ is said *locally finite* if for any compact subset $C \subset E$, $A \cap C$ is of finite cardinal. We equip E with its Borel σ -field \mathcal{E} .

Let \mathbb{Z} , \mathbb{N} and \mathbb{N}^* denote the sets of integers, of non-negative integers and of positive integers, respectively. We denote for any $x, y \in \mathbb{R}$, $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$ and $x^+ = x \vee 0$.

Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, furnished with the measurable bijective flow θ (denote θ^{-1} , its measurable inverse). Suppose that \mathbf{P} is stationary and ergodic under θ , i.e. for all $\mathcal{A} \in \mathcal{F}$, $\mathbf{P}[\theta^{-1}\mathcal{A}] = \mathbf{P}[\mathcal{A}]$ and all \mathcal{A} that is θ -invariant (i.e. such that $\theta\mathcal{A} = \mathcal{A}$) is of probability 0 or 1. Note that according to these axioms, all θ -contracting event (such that $\mathbf{P}[\mathcal{A}^c \cap \theta^{-1}\mathcal{A}] = 0$) is of probability 0 or 1. We denote for all $n \in \mathbb{N}$, $\theta^n = \theta \circ \theta \circ \dots \circ \theta$ and $\theta^{-n} = \theta^{-1} \circ \theta^{-1} \circ \dots \circ \theta^{-1}$. Except when explicitly mentioned, throughout all the random variables (r.v.'s for short) are defined on $(\Omega, \mathcal{F}, \mathbf{P})$. Under such conditions, the quadruple $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$ is termed *stationary ergodic dynamical system*.

We denote $\mathcal{M}(E)$ the set of measurable mappings from E into itself. For any $\mathcal{M}(E)$ -valued r.v. F , for any $x \in E$, let $F_\omega(x)$ be the image of x through F for the sample ω . For any $f \in \mathcal{M}(E)$, and any subset $B \subset E$, we denote $f(B) = \{f(x); x \in B\}$, and accordingly for any $\mathcal{M}(E)$ -valued r.v. F and all sample ω , $F_\omega(B) = \{F_\omega(x); x \in B\}$.

Let φ be a $\mathcal{M}(E)$ -valued r.v.. For all E -valued r.v. X , let $\{X_{X,n}\}_{n \in \mathbb{N}}$ be the *stochastic recursion* initiated by X and driven by φ , i.e., such that \mathbf{P} -a.s.,

$$\begin{cases} X_{X,0} &= X; \\ X_{X,n+1} &= \varphi \circ \theta^n(X_{X,n}), \text{ for all } n \in \mathbb{N}. \end{cases}$$

Define for all sample ω , all n and $x \in E$,

$$\Phi_\omega^n(x) = X_{x,n}(\theta^{-n}\omega) = \varphi_{\theta^{-1}\omega} \circ \varphi_{\theta^{-2}\omega} \circ \dots \circ \varphi_{\theta^{-n}\omega}(x).$$

The r.v. $\Phi^n(x)$ represents the value of the recursion driven at time 0 when starting at the iteration $-n$ from the deterministic value x . In other words,

$$\Phi_\omega^n(x) = X_{x,n} \circ \theta^{-n}.$$

We investigate the existence of a stationary version of the sequence $\{X_{X,n}\}_{n \in \mathbb{N}}$, i.e. such that $X_{X,n} = X \circ \theta^n$ for all $n \in \mathbb{N}$. Then it is easily seen that the r.v. X solves the functional equation

$$(1) \quad X \circ \theta = \varphi(X) \text{ a.s..}$$

The existence of a solution to (1) on the original probability space is not granted in general, without further assumptions on φ . We aim to construct an extension of the probability space, on which a solution exists.

3. AN EXISTENCE RESULT

Let us assume throughout this section that the couple (Ω, \mathcal{F}) is Polish (i.e. Ω is Polish and \mathcal{F} is a sub- σ -algebra of the Borel σ -algebra of Ω). Under certain conditions, the existence of an extension on which (1) admits a solution, is granted by Anantharam and Konstantopoulos's Theorem (see [1, 2]). This result, which identifies the probability measure on the extension as a weak limit, strongly relies on the property of tension of the embedded sequence of random variables. The latter holds, in particular, under the following domination assumption.

(H1) For some $\mathcal{M}(E)$ -valued r.v. ψ ,

- for all $x \in E$, $0_E \preceq \varphi(x) \preceq \psi(x)$, \mathbf{P} -a.s.;
- ψ is \mathbf{P} -a.s. \preceq -non decreasing and continuous;
- the following recursion admits at least one E -valued solution:

$$(2) \quad Y \circ \theta = \psi(Y).$$

We have the following result.

Proposition 1. Suppose that (H1) holds, and that either one of the two following conditions holds:

- (H2) φ is \mathbf{P} -a.s. continuous;
- (H3) φ admits a.s. a finite number of discontinuities, and there exists a locally finite subset $0_E \in L \subset E$ that is \mathbf{P} -a.s. stable by φ .

Then, there exists an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\theta})$ of $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$, that is such that

- $\tilde{\mathbf{P}}$ is a $\tilde{\theta}$ -invariant probability on $\tilde{\Omega}$ having Ω -marginal \mathbf{P} ,
- there exists a $E \times \mathcal{M}(E)$ -valued r.v. $(\tilde{X}, \tilde{\varphi})$ defined on $\tilde{\Omega}$ by (3), such that the Ω -marginal of $\tilde{\varphi}$ is the distribution of φ , and such that

$$\tilde{X} \circ \tilde{\theta} = \tilde{\varphi}(\tilde{X}), \tilde{\mathbf{P}}\text{-a.s..}$$

Proof. This result is a consequence of Theorem 1 in [1], whose hypothesis are completed in [2]. Define

- $\tilde{\Omega} := \Omega \times E$,
- $\tilde{\mathcal{F}} := \mathcal{F} \otimes \mathcal{E}$,
- for all $(\omega, x) \in \tilde{\Omega}$, $\tilde{\theta}(\omega, x) = (\theta\omega, \varphi_\omega(x))$.

As an immediate consequence of Loynes's Theorem for stochastic recursions ([11, 4]), there exists a solution, say Y_∞ , to (2). The r.v. Y_∞ is given by the a.s. limit of the sequence $\{Y_{0_E,n} \circ \theta^{-n}\}_{n \in \mathbb{N}}$ (see section 4.3 below). Note, that Y_∞ may in general be improper (i.e. valued in some $\bar{E} \supset E$). However, Y_∞ is \preceq -minimal among all the solutions of (2) (again, see 4.3), and the last assertion of (H1) entails that Y_∞ is E -valued.

In particular, the sequence $\{Y_{0_E,n}\}_{n \in \mathbb{N}}$ tends weakly to Y_∞ . It is thus tight (Prohorov's Lemma): for all $\varepsilon > 0$, there exists a compact subset K_ε of E such that for all $n \in \mathbb{N}$,

$$\mathbf{P}[Y_{0_E,n} \in K_\varepsilon] \geq 1 - \varepsilon.$$

Thus, as E is Lattice-ordered, there exists $M_\varepsilon \in E$ s.t.

$$\mathbf{P}[Y_{0_E,n} \preceq M_\varepsilon] \geq 1 - \varepsilon.$$

In view of the first assertion of (H1), an immediate induction shows that

$$X_{0_E,n} \preceq Y_{0_E,n}, n \in \mathbb{N} \text{ a.s.,}$$

so that

$$\mathbf{P}[X_{0_E,n} \preceq M_\varepsilon] \geq 1 - \varepsilon, n \in \mathbb{N},$$

which shows the tightness of $\{Y_{0_E,n}\}_{n \in \mathbb{N}}$.

Remark that for all $n \in \mathbb{N}$, $\mathcal{A} \in \mathcal{F}$ and $\mathcal{B} \in \mathcal{E}$,

$$\mathbf{P} \otimes \delta_{0_E} [\bar{\theta}^{-n}(\mathcal{A} \times E)] = \mathbf{P} [\theta^{-n}\mathcal{A}] = \mathbf{P}[\mathcal{A}]$$

and

$$\mathbf{P} \otimes \delta_{0_E} [\bar{\theta}^{-n}(\Omega \times \mathcal{B})] = \mathbf{P} \otimes \delta_{0_E} \left[\left\{ (\omega, x) \in \bar{\Omega}; X_{x,n}(\omega) \in \mathcal{B} \right\} \right] = \mathbf{P}[X_{0_E,n} \in \mathcal{B}].$$

Hence, the probability distributions $\{(\mathbf{P} \otimes \delta_{0_E}) \circ \bar{\theta}^{-n}\}_{n \in \mathbb{N}}$ on $\bar{\Omega}$ have Ω -marginal \mathbf{P} and E -marginals, the distributions of $\{X_{0_E,n}\}_{n \in \mathbb{N}}$, which form a tight sequence. The sequence $\{(\mathbf{P} \otimes \delta_{0_E}) \circ \bar{\theta}^{-n}\}_{n \in \mathbb{N}}$ is thus tight. Therefore, any sub-sequential limit is a good candidate for $\bar{\mathbf{P}}$ provided that it is $\bar{\theta}$ -invariant. This property holds under either one of conditions (A1)-(A3) p.271-272 in [2]. First, under condition (H2), the shift $\bar{\theta}$ is continuous from $\Omega \times E$ into itself, which is condition (A1) in [2].

Let us now assume that (H3) holds. Define for all ω , $\{d_j(\omega)\}_{j \in J_\omega}$, the set of discontinuities of φ_ω and

$$\delta(\omega) = \inf \{ \|d_j(\omega) - d_k(\omega)\|; j, k \in J_\omega \}.$$

Let us define the following events.

$$\mathcal{D} = \{\text{Card } J < \infty\};$$

$$\text{For all } p \in \mathbb{N}^*, \mathcal{E}_p = \left\{ \delta < 2^{-(p-1)} \right\}.$$

Note that by hypothesis, $\mathbf{P}[\mathcal{D}] = 1$, and thus $\mathbf{P}[\delta > 0] = 1$. Fix $p \in \mathbb{N}^*$ and a sample $\omega \in \mathcal{D}$. For all $j \in J_\omega$, define $C_{\omega,p,j}$ as follows :

- (a) if for some $k \in J_\omega$, $\|d_j(\omega) - d_k(\omega)\| \leq 2^{-(p-1)}$, $C_{\omega,p,j}$ is the open bowl of center $d_j(\omega)$ and radius δ ;
- (b) otherwise, $C_{\omega,p,j}$ is the open bowl of center $d_j(\omega)$ and radius 2^{-p} ,

so that the bowls $C_{\omega,p,j}$, $j \in J_\omega$, don't intersect. We define finally

$$C_{\omega,p} = \bigcup_{j \in J_\omega} C_{\omega,p,j},$$

and aim to construct a continuous function $\varphi_{\omega,p}$ from E into itself, such that $\varphi_{\omega,p}$ coincides with φ_ω outside the open set $C_{\omega,p}$. For doing so, fix j and let $x \in C_{\omega,p,j}$. There exists y in the frontier $\hat{C}_{\omega,p,j}$ of $C_{\omega,p,j}$ such that for some $\eta < 1$,

$$x - d_j(\omega) = \eta \cdot (y - d_j(\omega)).$$

Then, we set

$$\begin{aligned} \varphi_{\omega,p,j}(x) &:= 2^p \|x - d_j(\omega)\| \cdot \varphi(y) \text{ in case (a),} \\ \varphi_{\omega,p,j}(x) &:= \frac{1}{\delta(\omega)} \|x - d_j(\omega)\| \cdot \varphi(y) \text{ in case (b).} \end{aligned}$$

The function $\varphi_{\omega,p,j}$, hence radially defined, is clearly continuous on the bowl. Defining now for all $x \in E$,

$$\varphi_{\omega,p}(x) = \begin{cases} \varphi_\omega(x) & \text{if } x \notin C_{\omega,p} \\ \varphi_{\omega,p,j}(x) & \text{if } x \in C_{\omega,p,j}, \end{cases}$$

we obtain the desired function.

Finally, define the family of shifts $\bar{\theta}_p$, $p \geq 1$ for all $(\omega, x) \in \bar{\Omega}$ by

$$\bar{\theta}_p(\omega, x) = (\theta\omega, \varphi_{\omega,p}(x)).$$

The $\bar{\theta}_p$, $p \geq 1$ are then continuous from $\bar{\Omega}$ into itself. Now fix again $p \geq 1$. It is clear from (H3) that for all $i \geq 1$, $\Phi^i(0_E) \in L$ a.s., hence

$$\mathbf{P}[\{\omega; \Phi_\omega^i(0_E) \in C_{\omega,p}\} \cap \mathcal{D}] \leq \mathbf{P}[\{\omega; L \cap C_{\omega,p} \neq \emptyset\} \cap \mathcal{D}].$$

Finally, set

$$\mathcal{U}_p = \{(\omega, x) \in \bar{\Omega}; x \in C_{\omega,p}\}.$$

Then, \mathcal{U}_p is an open subset of $\bar{\Omega}$, and in view of the latter inequality,

$$\begin{aligned} \lim_{p \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} (\mathbf{P} \otimes \delta_x) \circ \bar{\theta}^{-i}(\mathcal{U}_p) \\ \leq \lim_{p \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \mathbf{P}[\{\Phi^i(0_E) \in C_p\} \cap \mathcal{D}] + \mathbf{P}[\bar{\mathcal{D}}] \\ \leq \lim_{p \rightarrow \infty} \mathbf{P}[\{L \cap C_p \neq \emptyset\} \cap \mathcal{D}] \\ \leq \lim_{p \rightarrow \infty} \mathbf{P}\left[\left\{\bigcup_{j \in J} L \cap C_{p,j} \neq \emptyset\right\} \cap \mathcal{D} \cap \left\{2^{-(p-1)} < \delta\right\}\right] \\ = 0, \end{aligned}$$

since L is locally finite. Consequently, Assumption (A3) p.272 of [2] is satisfied. Hence, from Theorem 1 of [1], there exists a $\bar{\theta}$ -invariant probability $\bar{\mathbf{P}}$ on $\Omega \times \mathbb{R}$ whose Ω -marginal is \mathbf{P} , given by any sub-sequential limit of $\{(\mathbf{P} \otimes \delta_{0_E}) \circ \bar{\theta}^{-n}\}_{n \in \mathbb{N}}$.

Now, define on $\bar{\Omega}$ the random variables

$$(3) \quad \bar{X}(\omega, x) := x, \quad \bar{\varphi}_{\omega,x} := \varphi_\omega.$$

We then have that

$$(4) \quad \bar{X} \circ \bar{\theta}(\omega, x) = \varphi_\omega(x) = \bar{\varphi}_{\omega,x}(x) = \bar{\varphi}_{\omega,x}(\bar{X}(\omega, x)), \bar{\mathbf{P}}\text{-a.s.,}$$

hence \bar{X} is a proper solution to (1) on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{P}}, \bar{\theta})$. \square

4. EXPLICIT CONSTRUCTION

We present the main result of this work. Under certain conditions, we can construct explicitly an extension solving equation (1). For doing so, we follow an argument related to that developed by Flipo [8] and Neveu [13] for the recursion describing the workload of a loss queueing system G/G/1/1.

We start with a random set G satisfying

$$(5) \quad \varphi_\omega(G_\omega) \subseteq G_{\theta\omega}, \text{ a.s.,}$$

which is checked *e.g.* by $G \equiv E$, or any deterministic set that is a.s. stable by φ . Now denote for all $n \in \mathbb{N}^*$

$$(6) \quad H_\omega^n := \Phi_\omega^n(G_{\theta^{-n}\omega}),$$

the set of all possible values of the recursion driven by φ at 0, when letting the value at $-n$ vary over the set $G_{\theta^{-n}\omega}$. Let us first remark that

Lemma 1. *The sequence of random sets $\{H_\omega^n\}_{n \in \mathbb{N}}$ decreases for inclusion:*

$$(7) \quad G \supseteq H^1 \supseteq H^2 \supseteq \dots \supseteq H^n \supseteq \dots \quad \text{a.s.}$$

Proof. That

$$H_\omega^1 = \varphi_{\theta^{-1}\omega}(G_{\theta^{-1}\omega}) \subseteq G_\omega, \text{ a.s.,}$$

simply follows from (5). Now, let $n \in \mathbb{N}^*$. We have a.s. for all $x \in H_\omega^{n+1}$, that for some $y \in G_{\theta^{-(n+1)}\omega}$,

$$x = \Phi_\omega^{n+1}(y) = \Phi_\omega^n(\varphi_{\theta^{-(n+1)}\omega}(y)).$$

since $y \in G_{\theta^{-(n+1)}\omega}$, we have that $\varphi_{\theta^{-(n+1)}\omega}(y) \in G_{\theta^{-n}\omega}$ in view of (5), hence $x \in H_\omega^{n+1}$. \square

We can thus define, a.s.,

$$(8) \quad H_\omega = \lim_{n \rightarrow \infty} H_\omega^n = \bigcap_{n \geq 1} H_\omega^n \subseteq G_\omega.$$

Lemma 2. *Assume that (5), and the following condition hold:*

The random set H defined by (8) is such that

$$(9) \quad \mathbf{P}[H \text{ is finite and non-empty}] > 0.$$

Then, the mapping φ is bijective from H to $H \circ \theta$, a.s.. The r.v. $\text{Card}H$ is thus deterministic, denoted by c .

Proof. Take a sample ω in the event

$$\mathcal{C} := \{H \text{ is finite and non-empty}\}.$$

For any $x \in H_\omega$, for all $n \geq 1$, there exists $y_n \in G_{\theta^{-n}\omega}$ such that $x = \Phi_\omega^n(y_n)$. Therefore,

$$\varphi_\omega(x) = \varphi_\omega \circ \varphi_{\theta^{-1}\omega} \circ \dots \circ \varphi_{\theta^{-n}\omega}(y_n) = \Phi_{\theta\omega}^{n+1}(y_n),$$

where $y_n \in G_{\theta^{-(n+1)}\omega} = G_{\theta^{-(n+1)}\theta\omega}$. This is true for all $n \geq 1$, hence $\varphi_\omega(x)$ belongs to the set

$$\bigcap_{n \geq 2} \Phi_{\theta\omega}^n(G_{\theta^{-n}\theta\omega}) = \bigcap_{n \geq 2} H_{\theta\omega}^n = H_{\theta\omega},$$

so that φ_ω maps H_ω onto $H_{\theta\omega}$. Consequently,

$$(10) \quad \begin{aligned} \mathcal{C} &\subset \mathcal{C} \cap \{\varphi(H) \subset H \circ \theta\} \\ &\subset \theta^{-1}\mathcal{C}. \end{aligned}$$

Hence, \mathcal{C} is θ -contracting, and then almost sure in virtue of (9). So is the event on the r.h.s. of (10), thus

$$0 < \text{Card } H \circ \theta \leq \text{Card } H < \infty, \text{ a.s..}$$

But

$$\mathbf{P}[\text{Card } H \circ \theta < \text{Card } H] > 0$$

would then imply that

$$\mathbf{E}[(\text{Card } H) \circ \theta - \text{Card } H] < 0,$$

a contradiction to the Ergodic Lemma ([4], Lemma 2.2.1). Therefore,

$$(\text{Card } H) \circ \theta = \text{Card } H \text{ a.s.,}$$

which shows that $\text{Card } H$ is deterministic, say equal to c a.s..

Now, to check that φ is a.s. surjective, fix a sample ω on the almost sure event

$$\{\text{Card } H = c\} \cap \theta^{-1}\{\text{Card } H = c\},$$

and $y \in H_{\theta\omega}$. In particular, for any $n \geq 1$, for some

$$x_{n+1} \in G_{\theta^{-(n+1)}\theta\omega} = G_{\theta^{-n}\omega},$$

we have that

$$y = \Phi_{\theta\omega}^{n+1}(x_{n+1}) = \varphi_\omega(y_n),$$

where

$$y_n = \varphi_{\theta^{-1}\omega} \circ \dots \circ \varphi_{\theta^{-n}\omega}(x_{n+1}) = \Phi_\omega^n(x_{n+1}) \in H_\omega^n.$$

Hence,

$$y_n \in \bigcap_{n \geq 1} \varphi_\omega(H_\omega^n) = \varphi \left(\bigcap_{n \geq 1} H_\omega^n \right),$$

where the last equality follows from Lemma 1. Therefore, φ_ω is surjective from H_ω into $H_{\theta\omega}$, and hence bijective since these two sets have the same cardinal. \square

We are now in position to construct an enrichment of the original probability space $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$ on which the existence of a solution to (1) is granted.

Proposition 2. *Suppose that (5) and (9) hold true. Then, the quadruple $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\theta})$ defines a stationary dynamical system:*

- $\tilde{\Omega} = \{(\omega, x) \in \Omega \times E; x \in H_\omega\};$
- $\tilde{\mathcal{F}}$ is the trace of $\mathcal{F} \otimes \mathcal{E}$ on $\tilde{\Omega}$, i.e.

$$\tilde{\mathcal{F}} = \left\{ \tilde{\mathcal{A}} := \left\{ (\omega, x) \in \Omega \times E; \omega \in \mathcal{A}, x \in \mathcal{B} \cap H_\omega, \right. \right.$$

$$\left. \left. \text{where } \mathcal{A} \in \mathcal{F} \text{ and } \mathcal{B} \in \mathcal{E} \right\} \right\}.$$

- For all $\tilde{\mathcal{A}} \in \tilde{\mathcal{F}}$ of the above form,

$$\tilde{\mathbf{P}}[\tilde{\mathcal{A}}] = \frac{1}{c} \int_{\Omega} \mathbf{1}_{\tilde{\mathcal{A}}}(\omega) \text{Card}(H_\omega \cap \mathcal{B}) d\mathbf{P}(\omega);$$

- For all $(\omega, x) \in \tilde{\Omega}$, $\tilde{\theta}(\omega, x) = (\theta\omega, \varphi_\omega(x)).$

Proof. To check this, first remark that $\tilde{\theta}$ defines an automorphism of $\tilde{\Omega}$ in view of Lemma 2. On another hand, $\tilde{\mathbf{P}}$ defines a probability measure, since it clearly is a σ -finite measure, that is such that

$$\tilde{\mathbf{P}}[\tilde{\Omega}] = \frac{1}{c} \int_{\Omega} \mathbf{1}_{\Omega}(\omega) \text{Card}(H_{\omega} \cap E) d\mathbf{P}(\omega) = 1.$$

Notice as well that $\tilde{\mathbf{P}}$ has Ω -marginal \mathbf{P} since for all $\mathcal{A} \in \mathcal{F}$,

$$(11) \quad \tilde{\mathbf{P}}[\mathcal{A} \times E] = \frac{1}{c} \int_{\Omega} \mathbf{1}_{\mathcal{A}}(\omega) \text{Card}(H_{\omega} \cap E) d\mathbf{P}(\omega) = \mathbf{P}[\mathcal{A}].$$

Now, fix $\tilde{\mathcal{A}} := \{(\omega, x) \in \tilde{\Omega}; \omega \in \mathcal{A}, x \in \mathcal{B} \cap H_{\omega}\} \in \tilde{\mathcal{F}}$. Then, remarking that $\tilde{\theta}(\omega, x) \in \tilde{\mathcal{A}}$ amounts to $\theta\omega \in \mathcal{A}$ and $\varphi_{\omega}(x) \in \mathcal{B} \cap H_{\theta\omega}$, we have that

$$\begin{aligned} \tilde{\mathbf{P}}[\tilde{\theta}^{-1}\tilde{\mathcal{A}}] &= \int \int_{\tilde{\Omega}} \mathbf{1}_{\theta^{-1}\mathcal{A}}(\omega) \mathbf{1}_{(\varphi_{\omega})^{-1}(\mathcal{B} \cap H_{\theta\omega})}(y) d\tilde{\mathbf{P}}(\omega, y) \\ &= \frac{1}{c} \int_{\Omega} \mathbf{1}_{\theta^{-1}\mathcal{A}}(\omega) \text{Card}\left((\varphi_{\omega})^{-1}(\mathcal{B} \cap H_{\theta\omega}) \cap H_{\omega}\right) d\mathbf{P}(\omega). \end{aligned}$$

But in view of Lemma 2,

$$\begin{aligned} \text{Card}\left((\varphi_{\omega})^{-1}(\mathcal{B} \cap H_{\theta\omega}) \cap H_{\omega}\right) &= \text{Card}\left((\varphi_{\omega})^{-1}(\mathcal{B} \cap H_{\theta\omega})\right) \\ &= \text{Card}(\mathcal{B} \cap H_{\theta\omega}), \end{aligned}$$

so by θ -invariance of \mathbf{P} ,

$$\begin{aligned} \tilde{\mathbf{P}}[\tilde{\theta}^{-1}\tilde{\mathcal{A}}] &= \frac{1}{c} \int_{\Omega} \mathbf{1}_{\mathcal{A}}(\theta\omega) \text{Card}(\mathcal{B} \cap H_{\theta\omega}) d\mathbf{P}(\omega) \\ &= \frac{1}{c} \int_{\Omega} \mathbf{1}_{\mathcal{A}}(\omega) \text{Card}(\mathcal{B} \cap H_{\omega}) d\mathbf{P}(\omega) \\ &= \tilde{\mathbf{P}}[\tilde{\mathcal{A}}], \end{aligned}$$

which first shows the measurability of $\tilde{\theta}^{-1}\tilde{\mathcal{A}}$, and second, the $\tilde{\theta}$ -invariance of $\tilde{\mathbf{P}}$. The proof is complete. \square

The quadruple $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\theta})$ is an enrichment of $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$: the first space is projected onto the second one by the mapping

$$f: \begin{cases} \tilde{\Omega} & \longrightarrow \Omega \\ (\omega, x) & \longmapsto \omega, \end{cases}$$

and for all $\mathcal{A} \in \mathcal{F}$,

$$\tilde{\mathbf{P}} \circ f^{-1}[\mathcal{A}] = \frac{1}{c} \int_{\Omega} \mathbf{1}_{\mathcal{A}}(\omega) \text{Card}(H_{\omega} \cap E) d\mathbf{P}(\omega) = \mathbf{P}[\mathcal{A}]$$

and

$$f \circ \tilde{\theta} \circ f^{-1}(\mathcal{A}) = \{f(\theta\omega, \varphi_{\omega}(x)); \omega \in \mathcal{A}, x \in H_{\omega}\} = \theta\mathcal{A}.$$

Let now \tilde{X} (resp. $\tilde{\varphi}$) be the restriction on $\tilde{\Omega}$ of the r.v. \tilde{X} (resp. $\tilde{\varphi}$) defined in (3), that is,

$$\begin{aligned} \tilde{X}(\omega, x) &= x, \tilde{\mathbf{P}} - \text{a.s.}, \\ \tilde{\varphi}_{\omega, x}(y) &= \varphi_{\omega}(y) \text{ for all } y \in E, \tilde{\mathbf{P}} - \text{a.s.} \end{aligned}$$

Then, as in (4),

$$(12) \quad \tilde{X} \circ \tilde{\theta} = \tilde{\varphi}(\tilde{X}), \tilde{\mathbf{P}} - \text{a.s.},$$

thus \tilde{X} is a solution to (1) on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\theta})$. We have proven the following result.

Theorem 1. *If some random set G satisfies (5) and (9), there exists a stationary extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\theta})$ of $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$, given in Proposition 2, on which the equation (1) admits a solution \tilde{X} , given by (12).*

4.1. Resolution on the original space. Assume in this sub-section that (5) holds together with (9). The Ergodicity of the dynamical system obtained in Theorem 1 is not a by-product of the construction, as easily understood, and similarly to that in [8, 13]. Notice nevertheless that the invariant sigma-field is easy to identify: let

$$\mathcal{I} = \{(\omega, x); \omega \in \mathcal{A}; x \in I_\omega\}$$

be a $\tilde{\theta}$ -invariant event of $\tilde{\mathcal{F}}$. Then, as

$$\tilde{\theta}^{-1} \mathcal{I} = \left\{ (\omega, x); \omega \in \theta^{-1} \mathcal{A}; x \in (\varphi_\omega)^{-1}(I_{\theta\omega}) \right\},$$

$\mathcal{I} = \tilde{\theta}^{-1} \mathcal{I}$ amounts to

$$\begin{cases} \tilde{\theta}^{-1} \mathcal{A} = \mathcal{A} \\ \forall \omega \in \mathcal{A}, I_{\theta\omega} = \varphi_\omega(I_\omega). \end{cases}$$

Then, in view of the ergodicity of θ , all invariant of $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\theta})$ can be written, up to a $\tilde{\mathbf{P}}$ -negligible event, as

$$(13) \quad \mathcal{I} = \{(\omega, x) \in \tilde{\Omega}; x \in I_\omega\},$$

where

$$(14) \quad I_{\theta\omega} = \varphi_\omega(I_\omega), \mathbf{P} - \text{a.s.}$$

A simple criterion of existence for a proper solution of (1) on the original space is then given in the following Lemma.

Lemma 3. *There exists a bijection between the two following sets:*

$$\begin{aligned} \mathcal{J} &:= \left\{ E\text{-valued solutions of (1) on } (\Omega, \mathcal{F}, \mathbf{P}, \theta) \text{ such that } \mathbf{P}[X \in G] > 0 \right\} \\ &\longleftrightarrow \mathcal{K} := \left\{ \tilde{\theta}\text{-invariant sets of the form (13), s.t. Card } I = 1 \text{ a.s.} \right\}. \end{aligned}$$

Proof. Let X be an element of \mathcal{J} , $I_\omega = \{X(\omega)\}$, a.s., and

$$\mathcal{I} = \{(\omega, X(\omega)); \omega \in \Omega\}.$$

We first have to check that $\mathcal{I} \in \tilde{\mathcal{F}}$, i.e. that $X \in H$, a.s.. Remark that $\{X \in G\}$ is θ -contracting, since a.s., whenever $X(\omega) \in G_\omega$,

$$X(\theta\omega) = \varphi_\omega(X(\omega)) \in \varphi_\omega(G_\omega) \subseteq G_{\theta\omega}.$$

This event, which has a positive probability, is thus almost sure. Hence, by θ -invariance

$$\mathbf{P} \left[\bigcap_{n \geq 1} \theta^n \{X \in G\} \right] = 1,$$

therefore $X \in H$, a.s.. On another hand,

$$I_{\theta\omega} = \{X(\theta\omega)\} = \{\varphi_\omega(X(\omega))\} = \varphi_\omega(I_\omega), \text{ a.s.,}$$

so that $\mathcal{I} \in \mathcal{K}$ in view of (14).

Conversely, given $\mathcal{I} = \{(\omega, i(\omega)) \in \tilde{\Omega}\} \in \mathcal{K}$, it is easily seen that the r.v. defined on $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$ by $X(\omega) = i(\omega)$ is a E -valued solution to (1). Moreover, a.s.,

$$X(\omega) = i(\omega) \in H_\omega \subseteq G_\omega,$$

so the r.v. X is an element of \mathcal{J} . □

Then, readily

Corollary 1. *There exists a solution to (1) on the original probability space, possibly taking values in G , iff \mathcal{K} is non-empty, which is the unique such solution iff \mathcal{K} is reduced to a singleton. In particular, there exists a unique such solution whenever $c = 1$.*

We now address the closely related question of convergence of the embedded recursive sequence. Remember, that we denote for E -valued r.v. X ,

$$X_{X,n}(\omega) = \varphi_{\theta^{n-1}\omega} \circ \dots \circ \varphi_{\omega}(X(\omega)), \text{ a.s.,}$$

the value after n steps of the recursion initiated by X and driven by φ .

Corollary 2. *Whenever \mathcal{K} is non-empty, any element of \mathcal{J} is the weak limit of some sequence $\{X_{Y,n}\}_{n \in \mathbb{N}}$, where Y is some E -valued r.v. such that $Y \in G$, a.s.. If moreover*

$$(15) \quad \mathbf{P}[\text{For some } N \text{ the set } H^N \text{ has a finite cardinal}] > 0,$$

the latter convergence holds with strong backwards coupling.

Proof. It is clear by the very definition of H that for any $X \in \mathcal{J}$, for some r.v. Y such that $Y \in G$, a.s., the sequence $\Phi^n(Y) = \{X_{Y,n} \circ \theta^{-n}\}_{n \in \mathbb{N}}$ converges a.s. to X . Hence, by θ -invariance the sequence $\{X_{Y,n}\}_{n \in \mathbb{N}}$ tends in distribution to X .

Now, on the event in (15), for any $y \in H_{\theta\omega}^{N(\omega)+1}$, there exists $x \in G_{\theta^{-N(\omega)}\omega}$ such that

$$y = \varphi_{\omega}(\varphi_{\theta^{-1}\omega} \circ \dots \circ \varphi_{\theta^{-N(\omega)}\omega}(x)),$$

so that $y \in \varphi_{\omega}(H_{\omega}^{N(\omega)})$. Hence,

$$H_{\theta\omega}^{N(\omega)+1} \subset \varphi_{\omega}(H_{\omega}^{N(\omega)}),$$

which implies that the event in (15) is θ -contracting (taking $N(\theta\omega) = N(\omega) + 1$), and hence almost sure whenever (15) holds. Therefore, in that case there exists a.s. an integer $N' \geq N$ such that for all $n \geq N'$, $H^n = H$. Hence, for any Y as above, $X = \Phi^n(Y) = X_{Y,n} \circ \theta^{-n}$ a.s. for all $n \geq N'$. In other words, there is strong backwards coupling between the sequences $\{X_{Y,n}\}_{n \in \mathbb{N}}$ and $\{X \circ \theta^n\}_{n \in \mathbb{N}}$ with coupling time N' . □

4.2. Applications. We present hereafter several cases in which Theorem 1 applies.

Proposition 3. *Conditions (5) and (9) are met, and then Theorem 1 applies, in the following cases:*

- (i) *Some deterministic finite subset F of E is a.s. stable by φ ;*
- (ii) *The random map φ is itself a.s. continuous and \preceq -nondecreasing;*
- (iii) *(H1) holds and for some solution Y to (2),*

$$(16) \quad \mathbf{P}[Y \leq 0] > 0.$$

In this case, a unique E -valued solution X to (1) exists on the original probability space, to which all sequences $\{X_{Z,n}\}_{n \in \mathbb{N}}$, $Z \preceq Y$ a.s., converge with strong backwards coupling;

- (iv) *(H1) and (H3) hold;*

- (v) For some collection \mathcal{G} of E -valued r.v.'s, there exists an integer p , a finite random set B and an event \mathcal{B} of positive probability, such that for all $Z \in \mathcal{G}$ and all $n \geq p$,

$$X_{Z,n} \in B \circ \theta^n \text{ on } \theta^{-n}\mathcal{B}.$$

If additionally, $\mathbf{P}[\text{Card } B = 1] > 0$, a solution X to (1) exists on the original space, to which all sequences $\{X_{Z,n}\}_{n \in \mathbb{N}}$, $Z \in \mathcal{G}$, converge with strong backwards coupling.

- Proof.* (i) Take $G = F$ a.s., so (5) and (9) trivially hold.
(ii) The recursion driven by φ hence satisfies to Loynes's Theorem. See subsection 4.3 below.
(iii) Suppose that (H1) holds, and let Y be an arbitrary E -valued solution to (2). Set

$$G = \llbracket 0, Y \rrbracket \text{ a.s..}$$

Then, a.s. for all $y \in \varphi_\omega(G_\omega)$, $y = \varphi_\omega(x)$ for some $x \in E$ such that $x \preceq Y(\omega)$. But in view of (H1),

$$(17) \quad y \preceq \psi_\omega(x) \preceq \psi_\omega(Y(\omega)) = Y(\theta\omega),$$

so that $y \in G_{\theta\omega}$. Hence G satisfies to (5).

Now, as a consequence of Birkhoff's Theorem, (16) implies that a.s., for some $N(\omega)$, $Y(\theta^{-N(\omega)}\omega) = 0_E$. Hence, $G_{\theta^{-N(\omega)}\omega} = \{0_E\}$ and

$$H_\omega^{N(\omega)} = \left\{ \Phi_\omega^{N(\omega)}(0_E) \right\}.$$

Therefore, (9) holds, and $c = 1$. In particular, in view of Corollary 1, a unique solution X to (1) exists on the original probability space, that is such that

$$\mathbf{P}[X \in G] = \mathbf{P}[X \preceq Y] > 0.$$

But on the event $\{X \preceq Y\}$, again in view of (H1),

$$X \circ \theta = \varphi(X) \preceq \psi(X) \preceq \psi(Y) = Y \circ \theta.$$

This event is thus θ -contracting, and hence almost sure. This shows that X is the only E -valued solution to (1). The strong backwards coupling property readily follows from Corollary 2.

- (iv) Suppose now that (H3) holds additionally to (H1). Let Y be a proper solution to (2) and $L \subset E$ be a locally finite subset of E that is a.s. stable by φ . Thus, as in (iii), (5) is clearly met as well by

$$G := \llbracket 0, Y \rrbracket \cap L \text{ a.s..}$$

Moreover, G is a.s. of finite cardinal in view of the locally-finiteness of L , so (9) holds true.

- (v) Set

$$(18) \quad G_\omega = \{Z(\omega); Z \in \mathcal{G}\}, \text{ a.s.,}$$

and let $Z \in \mathcal{G}$ and $Y = \varphi(Z) \circ \theta^{-1}$. Fix $n \geq p$ and $\omega \in \theta^{-n}\mathcal{B}$. Then, $\theta^{-1}\omega \in \theta^{-(n+1)}\mathcal{B}$, so

$$\begin{aligned} X_{Y,n}(\omega) &= \varphi_{\theta^{n-1}\omega} \circ \dots \circ \varphi_{\theta\omega} \circ \varphi_{\omega} (\varphi_{\theta^{-1}\omega} (Z(\theta^{-1}\omega))) \\ &= \varphi_{\theta^n(\theta^{-1}\omega)} \circ \dots \circ \varphi_{\theta(\theta^{-1}\omega)} \circ \varphi_{\theta^{-1}\omega} (Z(\theta^{-1}\omega)) \\ &= X_{Z,n+1}(\theta^{-1}\omega) \\ &\in B_{\theta^{n+1}(\theta^{-1}\omega)} = B_{\theta^{-n}\omega}. \end{aligned}$$

This is true on $\theta^{-n}\mathcal{B}$ for all $n \geq p$, so the r.v. $Y \in \mathcal{G}$. Hence,

$$\mathbf{P}[\theta\{\omega; \varphi_{\omega}(G_{\omega}) \subset G_{\theta\omega}\}] = 1,$$

which amounts to (5).

It remains to check (9). Let $\omega \in \mathcal{B}$ and $x_{\omega} \in G_{\theta^{-n}\omega}$. This means that for some r.v. $Z \in \mathcal{G}$, $x_{\omega} = Z(\theta^{-n}\omega)$. Hence, for all $n \geq p$, as $\theta^{-n}\omega \in \theta^{-n}\mathcal{B}$, we have that

$$\Phi_{\omega}^n(x_{\omega}) = X_{Z,n}(\theta^{-n}\omega) \in B_{\theta^n(\theta^{-n}\omega)} = B_{\omega}.$$

Therefore, on \mathcal{B} , $H \subset H^n \subset B$. As in the proof of Corollary 2 this implies, first, that H is finite on \mathcal{B} and second, that H is non-empty on \mathcal{B} since it coincides with H^n after a certain rank. Hence (9) holds since \mathcal{B} is assumed to have a positive probability.

Finally, on the event $\{\text{Card } B = 1\}$, $H = H^n = B$ for all $n \geq p$, so $c = 1$. Whenever this event is of positive probability, the latter is true a.s., so $c = 1$ and we can set $H_{\omega} = \{X(\omega)\}$, a.s.. Once again, the strong backwards coupling to X follows from Corollary 2. □

Whenever (H1) holds together with (H3), (iv) provides an alternative proof of Proposition 1. In fact, by $\tilde{\theta}$ -invariance of $\tilde{\mathbf{P}}$, the sequence of probability $\{\tilde{\mathbf{P}} \circ \tilde{\theta}^{-n}\}_{n \in \mathbb{N}}$ is tight since it is constant. So replacing $\mathbf{P} \otimes \delta_{0_E}$ by $\tilde{\mathbf{P}}$ (which has Ω -marginal \mathbf{P} - see (11)) in the proof of Proposition 1 would lead to the same extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\theta})$.

4.3. On Loynes's Theorem. Our construction allows us to capture Loynes' celebrated Theorem for monotonic recursions ([4, 11]). We assume here that φ is a.s. non-decreasing and continuous on E . Loynes's sequence is classically defined as $\{\Phi^n(0_E)\}_{n \in \mathbb{N}}$. It is routine to check that the latter is a.s. nondecreasing. Let a.s., Y , its supremum, that we assume to be E -valued. Then, by continuity,

$$(19) \quad \varphi(Y) = Y \circ \theta \text{ a.s..}$$

We set

$$G = \llbracket 0, Y \rrbracket \text{ a.s..}$$

Let $n \geq 1$. From (19), we have that

$$\Phi_{\omega}^n(Y(\theta^{-n}\omega)) = Y(\omega) \text{ a.s..}$$

Therefore, since Φ^n is a.s. non-decreasing (as easily seen by induction),

$$\begin{aligned} H_{\omega}^n &= \{\Phi_{\omega}^n(x); x \in \llbracket 0_E; Y(\theta^{-n}\omega) \rrbracket\} \\ &= \llbracket \Phi_{\omega}^n(0_E); \Phi_{\omega}^n(Y(\theta^{-n}\omega)) \rrbracket \\ &= \llbracket \Phi_{\omega}^n(0_E); Y(\omega) \rrbracket \text{ a.s..} \end{aligned}$$

As Y is the a.s. limit of Loynes's sequence, it readily follows that

$$H = \{Y\} \text{ a.s..}$$

Using Corollary 1, we obtain that the only solution Z to (1) on the original space such that $\mathbf{P}[Z \leq Y] > 0$ is the r.v. Y itself. Thus Y is the a.s. minimal solution, which is the exact statement of Loynes's Theorem.

4.4. Renovating events. Condition (v) of Proposition 3 can be rephrased in the following comprehensive terms: whatever the initial r.v. $X_0 = X$ in a given collection, after a deterministic rank N , the recursion is valued with positive probability in a finite range depending only upon the sample. We will give in Section 5 a concrete application of this result, which is, clearly, a generalization of the concept of *Renovating events* (see [6] and [4], p.115). In fact, we have expressed condition (v) in the form that better emphasizes this connexion. This will allow us to show readily that the typical existence and coupling result of Renovating events theory (Corollary 2.5.1 in [4]) is in fact a particular case of (v) of Proposition 3.

Let us briefly recall that a stationary sequence of events $\{\theta^{-n}\mathcal{A}\}_{n \in \mathbb{N}}$ (where \mathcal{A} is of positive probability) is termed sequence of renovating events of length $m \in \mathbb{N}^*$ for the recursion $\{X_n\}_{n \in \mathbb{N}}$ whenever for some E' -valued r.v. β (where E' is some auxilliary space), some deterministic mapping $\Psi : (E')^m \rightarrow E$, for all $n \geq m$,

$$(20) \quad X_n = \Psi(\beta \circ \theta^{n-m}, \dots, \beta \circ \theta^{n-2}, \beta \circ \theta^{n-1}) \text{ on } \theta^{-(n-m)}\mathcal{A}.$$

Now let \mathcal{Z} a collection of r.v.'s, for which we assume that all sequences $\{X_{Z,n}\}_{n \in \mathbb{N}}$, $Z \in \mathcal{Z}$, admit the same sequence of renovating events $\{\theta^{-n}\mathcal{A}\}_{n \in \mathbb{N}}$, with the same length m and same function Ψ . It is then straightforward that (v) holds. Take indeed $\mathcal{G} := \mathcal{Z}$, $\mathcal{B} := \theta^m\mathcal{A}$ and $p := m$. Then, for all $n \geq p$, $\theta^{-n}\mathcal{B} = \theta^{-(n-m)}\mathcal{A}$, so on this event,

$$X_{Z,n}(\omega) = \Psi(\beta \circ \theta^{n-m}, \dots, \beta \circ \theta^{n-2}, \beta \circ \theta^{n-1}) \text{ for all } Z \in \mathcal{Z}.$$

Therefore, condition (v) is satisfied when taking

$$B = \{\Psi(\beta \circ \theta^{-m}, \dots, \beta \circ \theta^{-2}, \beta \circ \theta^{-1})\}, \text{ a.s..}$$

In particular, $c = 1$, so there is a unique solution to (1) on the original probability space, to which all sequences $\{X_{Z,n}\}_{n \in \mathbb{N}}$, $Z \in \mathcal{Z}$ converge with strong backwards coupling. This is Borovkov and Foss's Theorem (see [6] and [4], Corollary 2.5.1).

5. THE LOSS QUEUE

The classical, but challenging problem of finding the stability region of the *Loss Queue* G/G/1/1 can be addressed in our framework. Consider a queueing system having one server and no waiting room, so that each customer is either immediately served (if the system is empty), or rejected upon arrival (if the server is busy). We assume that the input in this queue is of the G/G type, and work on the Palm space $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$ of the arrival process

$$\dots < T_{-2} < T_{-1} < T_0 = 0 < T_1 < \dots,$$

where T_n is the arrival time of the n th customer, denoted C_n . The stationary sequence of inter-arrival times $\{\xi_n\}_{n \in \mathbb{Z}} := \{T_{n+1} - T_n\}_{n \in \mathbb{Z}}$ is then compatible with θ , i.e. $\xi_n = \xi \circ \theta^n$ for all n . The service times $\{\sigma_n\}_{n \in \mathbb{Z}}$ requested by the customers form a sequence of marks of the arrival process, which is hence as well compatible with θ (see e.g. [4] for the Ergodic-theoretical representation of stationary queueing systems). We denote σ the generic service time, and assume that σ is a.s. non-negative, and ξ is a.s. positive.

Let W_n be the workload (i.e. the quantity of work in the system, in time unit) seen by C_n upon arrival. As easily checked, the workload sequence is a \mathbb{R}^+ -valued recursion driven by the random map

$$\varphi_\omega(x) = [x + \sigma(\omega)\mathbf{1}_{\{x=0\}} - \xi(\omega)]^+,$$

which is not a.s. non-decreasing and monotonic. Despite its simplicity, this model can not be handled by Loynes's framework. As a matter of fact, it is quite simple to exhibit examples for which uniqueness, and even existence of a solution to (1) don't hold (see [4], p.121 - and the examples hereafter). The existence of a stationary workload defined on $\Omega \times \mathbb{N}$, and a constructive scheme are presented in [8, 13], whereas the existence on $\Omega \times \mathbb{R}^+$ is proven in [1, 2] using the tightness approach, as developped in section 3. Hereafter, we use Theorem 1 to construct explicitly this solution, and relate it to those of [8, 13].

First, denote a.s.

$$(21) \quad A = \left\{ i > 0; \sigma \circ \theta^{-i} - \sum_{j=1}^i \xi \circ \theta^{-j} > 0 \right\};$$

$$(22) \quad \gamma = \sup A.$$

The set A thus contains all the absolute values of indexes of the customers possibly in the system at time 0, which are those who found an empty system upon arrival, and did not complete their service at 0.

Remark, that

Lemma 4. *The r.v. γ is a.s. finite. In particular, there exists an integer g such that*

$$(23) \quad g = \min \{ n > 0; \mathbf{P}[\gamma \leq n] > 0 \}.$$

Proof. It is a consequence of Birkhoff's Theorem that

$$\sigma \circ \theta^{-n} - \sum_{j=1}^n \xi \circ \theta^{-j} \xrightarrow[n \rightarrow \infty]{} -\infty \text{ a.s.,}$$

so there exists a.s. $N < +\infty$ such that the latter expression is non-positive for all $i \geq N$. In particular, a.s. $\gamma \leq N < +\infty$. For any n such that $\mathbf{P}[N = n] > 0$ (such integers exists since $N < +\infty$ a.s.), $\mathbf{P}[\gamma \leq n] > 0$, so g is well-defined. \square

In view of the above remark, on the event $\{\gamma \leq g\}$ the workload at time 0 is an element of the set

$$B := \left\{ \sigma \circ \theta^{-i} - \sum_{j=1}^i \xi \circ \theta^{-j}; i = 1, \dots, g \right\}.$$

In other words, for any E -valued r.v Z and for all $n \geq g$, $\Phi^n(Z \circ \theta^{-n}) \in B$ on $\{\gamma \leq g\}$, that is to say

$$W_{Z,n} \in B \circ \theta^n \text{ on } \theta^{-n}\{\gamma \leq g\}.$$

We are thus in the case (v) of Proposition 3 taking $G := \mathbb{R}^+$ a.s., $p := g$ and $\mathcal{B} = \{\gamma \leq g\}$. Theorem 1 thus applies to the workload sequence: there exists an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\theta})$ on which (1) admits a solution.

We aim to compare our extension to that presented in [8]. Let us briefly recall the construction proposed therein. Define almost surely, for all $i \in \mathbb{N}$,

$$\ell_\omega(i) = \begin{cases} i+1 & \text{if } C_{-i}, \text{ provided he found an empty system upon arrival,} \\ & \text{is still in service at } T_0-, \\ 0 & \text{else,} \end{cases}$$

and for all $n \geq 1$,

$$L_\omega^n(i) = \ell_{\theta^{-1}\omega} \circ \ell_{\theta^{-2}\omega} \circ \dots \circ \ell_{\theta^{-n}\omega}(i).$$

In words, $L^n(i)$ represents the index of the customers present in the system at T_0- when assuming that customer C_{-n-i} found an empty system upon arrival. Denoting then $\hat{H}_\omega^n = L_\omega^n(\mathbb{N})$ and $\hat{H}_\omega = \bigcap_{n \geq 1} \hat{H}_\omega^n$, one can show (see [8]), as in Lemma 2, that \hat{H} is an a.s. finite subset of \mathbb{N} having a deterministic cardinal. Hence, an enrichment $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbf{P}}, \hat{\theta})$ exists, that is defined similarly to that in Proposition 2, replacing φ by ℓ and H by \hat{H} . Moreover, (1) is solved on this extension by setting

$$\hat{X}(\omega, i) = \left[\sigma(\theta^{-i}\omega) - \sum_{j=1}^i \xi(\theta^{-j}\omega) \right]^+, \hat{\mathbf{P}} - \text{a.s.},$$

$$\hat{\phi}_{\omega, i} = \varphi_\omega, \hat{\mathbf{P}} - \text{a.s.}.$$

As shown in the next Lemma, $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbf{P}}, \hat{\theta})$ can be projected onto the extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\theta})$ constructed by Proposition 2.

Lemma 5. *The following mapping is a.s. surjective:*

$$F_\omega : \begin{cases} \hat{H}_\omega & \longrightarrow H_\omega \\ i & \longmapsto \Phi_\omega^i(0), \end{cases}$$

where $\Phi_\omega^0(0)$ is naturally set to 0.

Proof. Fix a sample ω , and let us first check that F_ω maps \hat{H}_ω onto H_ω . Let $j \in \hat{H}_\omega$. For all $n \geq 1$, there exists $i_n \in \mathbb{N}$ such that $j = L_\omega^n(i_n)$. In other words, for the sample ω , C_{-j} is in service just before time T_0 whenever C_{n+i_n} entered an empty system, hence

$$\Phi_\omega^{n+i_n}(0) = \Phi_\omega^j(0) = F_\omega(j).$$

Therefore, $F_\omega(j) \in \Phi_\omega^{n+i_n}(\mathbb{R}_+)$, so there exists $n' = n + i_n \geq n$ such that $F_\omega(j) \in H_\omega^{n'}$. This is true for all $n \geq 1$, hence $F_\omega(j) \in H_\omega$.

Now, to check that F_ω is surjective, take $x \in H_\omega$ and let for all $n \geq 1$, $x_n \in [0, Y(\theta^{-n}\omega)]$ be such that $x = \Phi_\omega^n(x_n)$. First, as shown above, there exists $j \in \{0, 1, \dots, \gamma\}$ such that

$$x = \Phi_\omega^j(0) = F_\omega(j).$$

Fix now $n \geq 1$. Then, assuming that for all $\tilde{n} \geq n$,

$$x_{\tilde{n}}(\omega) - \sum_{j=n+1}^{\tilde{n}} \xi(\theta^{-j}\omega) \geq 0$$

would contradict Birkhoff's Theorem (remember that $\mathbf{E}[\xi] > 0$). Then, there exists $\tilde{n} \geq n$ such that $x_{\tilde{n}}(\omega) - \sum_{j=n+1}^{\tilde{n}} \xi(\theta^{-j}\omega) < 0$, which means that either (i) $x_{\tilde{n}} = 0$ and the system was empty upon the arrival of $C_{-\tilde{n}}$ or (ii) $C_{-\tilde{n}}$ found a busy server upon arrival, having a residual workload of $x_{\tilde{n}}$, and the customer in service at that instant has left the system before the arrival of C_{-n} . In both cases, whenever $C_{-\tilde{n}}$ found a workload equal to $x_{\tilde{n}}$ upon arrival, there exists an index $\hat{n} \in \{n, n+1, \dots, \tilde{n}\}$ such that the system is empty at the arrival of $C_{-\hat{n}}$. In other words, $\Phi_\omega^{\tilde{n}}(x_{\tilde{n}}) = \Phi_\omega^{\hat{n}}(0)$.

As a consequence, there exists a non negative integer $i_n := \hat{n} - n$ such that

$$\Phi_\omega^j(0) = x = \Phi_\omega^{\tilde{n}}(x_{\tilde{n}}) = \Phi_\omega^{n+i_n}(0),$$

which amounts to say that $j = L_\omega^n(i_n)$. This is true for all $n \geq 1$, hence $j \in \hat{H}_\omega$, which concludes the proof. \square

We now introduce two simple examples (given in [4], p.122), in which existence or uniqueness of a stationary workload don't hold on the original probability space, and address the stability problem in our framework. We work on the following elementary ergodic dynamical system:

$$\begin{cases} \Omega &= \{\omega_1, \omega_2\}; \\ \mathcal{F} &= \mathcal{P}(\Omega); \\ \mathbf{P} &:= \text{uniform on } \Omega; \\ \theta &: \omega_1 \longleftrightarrow \omega_2. \end{cases}$$

Example 1

Set, say,

$$\begin{cases} \xi(\omega_1) &= \xi(\omega_2) = 1; \\ \sigma(\omega_1) &= 1, \sigma(\omega_2) =: y > 2. \end{cases}$$

We will only treat in detail the case where $y \notin \mathbb{N}$ and $\lfloor y \rfloor$ is an odd number. The other cases are analogous. Then, readily

$$\begin{aligned} A_{\omega_1} &= \{1, 3, 5, \dots, \lfloor y \rfloor\}, \gamma(\omega_1) = \lfloor y \rfloor; \\ A_{\omega_2} &= \{2, 4, 6, \dots, \lfloor y \rfloor - 1\}, \gamma(\omega_2) = \lfloor y \rfloor - 1. \end{aligned}$$

Let $i \in A_{\omega_1}$ and fix $n \geq 1$. Then, it is always possible to find an $x \in \mathbb{R}^+$ such that when assuming that the recursion equals x at the arrival of customer C_{-n} , C_{-i} is in service at time 0. Indeed,

- If n is odd,
 - If $n \equiv i \pmod{(\lfloor y \rfloor + 1)}$, say $n = i - 2 + p(\lfloor y \rfloor + 1)$, set

$$W_{-n} := x = 0.$$

Then C_{-n} is served, and

$$\begin{aligned} W_{-(i+(p-1)(\lfloor y \rfloor + 1)+1)} &= y - \left\{ (i + p(\lfloor y \rfloor + 1)) - (i + (p-1)(\lfloor y \rfloor + 1) + 1) \right\} \\ &= y - \lfloor y \rfloor > 0, \end{aligned}$$

whereas

$$W_{-(i+(p-1)(\lfloor y \rfloor + 1))} = [y - \{\lfloor y \rfloor + 1\}]^+ = 0.$$

Therefore, $C_{-(i+(p-1)(\lfloor y \rfloor + 1))}$ is served and by an immediate induction, all the customers C_{-k} , where $k \in \{i, \dots, n-1\}$ and $k \equiv i \pmod{(\lfloor y \rfloor + 1)}$ are served. In particular, C_{-i} is served, and is still in the system at 0. So

$$\Phi_{\omega_1}^n(x) = \sigma \circ \theta^{-i}(\omega_1) - \sum_{j=1}^i \xi \circ \theta^{-j}(\omega_1) = y - i;$$

- If $n \equiv i - 2\ell \pmod{(\lfloor y \rfloor + 1)}$ (say $n = i - 2\ell + p(\lfloor y \rfloor + 1)$), set

$$W_{-n} := x \in [\lfloor y \rfloor - 2\ell, \lfloor y \rfloor - 2\ell + 1].$$

Then,

$$\begin{aligned} W_{-(i+(p-1)(\lfloor y \rfloor + 1)+1)} &= x - \left\{ (i - 2\ell + p(\lfloor y \rfloor + 1)) - (i + (p-1)(\lfloor y \rfloor + 1) + 1) \right\} \\ &= x - \{\lfloor y \rfloor - 2\ell\} > 0, \end{aligned}$$

whereas

$$W_{-(i+(p-1)(\lfloor y \rfloor + 1))} = [x - \{\lfloor y \rfloor - 2\ell + 1\}]^+ = 0,$$

so $C_{-(i+(p-1)(\lfloor y \rfloor + 1))}$ is served, and as above, all C_{-k} where

$k \in \{i, \dots, (i + (p-1)(\lfloor y \rfloor + 1))\}$ and $k \equiv i \pmod{(\lfloor y \rfloor + 1)}$ are served. Here again, C_{-i} is thus served and $\Phi_{\omega_1}^n(x) = y - i$.

- if n is even, set $W_{-n} = x + 1$ for the different values of x set above, so that $W_{-(n-1)} = x$ for $n - 1$ odd and we can apply the above argument.

Therefore, in any case, for all $i \in A_{\omega_1}$ and all n , there exists $x \in \mathbb{R}^+$ such that $\Phi_{\omega_1}^n(x) = y - i$. This shows that $y - i \in H_{\omega_1}$ for all such i . On the other hand, provided some customer C_{-n} is served, where n is even and $n > \lfloor y \rfloor$, the service time of C_{-n} is 1, so $C_{-(n-1)}$ is served as well. Then $n - 1$ is odd with $n \equiv i - 2 \pmod{(\lfloor y \rfloor + 1)}$ for some $i \in A_{\omega_1}$, so as above C_{-i} is in service at 0. In particular, there is *always* a customer in service at time 0.

Therefore, for all $n \geq \lfloor y \rfloor + 1$,

$$\{y - i; i \in A_{\omega_1}\} \subset H_{\omega_1} \subset \Phi_{\omega_1}^n(\mathbb{R}^+) \subset \{y - i; i \in A_{\omega_1}\},$$

hence

$$H_{\omega_1} = \{y - 1, y - 3, y - 5, \dots, y - \lfloor y \rfloor\}.$$

Analogously, we can check that

$$H_{\omega_2} = \{y - 2, y - 4, y - 6, \dots, y - (\lfloor y \rfloor - 1), 0\}$$

(indeed the system may be empty at 0 for the sample ω_2 whenever $C_{-\lfloor y \rfloor + 1}$ is served). In particular, $c = \frac{\lfloor y \rfloor + 1}{2}$.

Now notice that $\varphi_{\omega_1}(y - i) = [y - (i + 1)]^+$ for all odd i such that $i \leq \lfloor y \rfloor$, whereas $\varphi_{\omega_2}(y - (i + 1)) = y - (i + 2)$ for all odd i , $i < \lfloor y \rfloor$ and $\varphi_{\omega_2}(0) = y - 1$. As a conclusion, recalling (13) we easily check that the invariant sigma-field of $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\theta})$ is $\{\emptyset, \tilde{\Omega}\}$. In particular, the set \mathcal{K} of Lemma 3 is empty: there is no solution on the original probability space.

Example 2

On the same probability space, define now

$$\begin{cases} \xi(\omega_1) &= \xi(\omega_2) = 1; \\ \sigma(\omega_1) &=: x, \sigma(\omega_2) =: y, \end{cases}$$

where x and y both belong to the open interval $(1, 2)$. Following the same lines as in Example 1, it is easily seen that

$$A_{\omega_1} = \{1\}; A_{\omega_2} = \{1\},$$

and

$$H_{\omega_1} = \{0, y - 1\}; H_{\omega_2} = \{0, x - 1\}.$$

It is then immediate that both events

$$\begin{aligned} \mathcal{J} &= \{(\omega_1, 0); (\omega_2, x - 1)\}, \\ \mathcal{J}' &= \{(\omega_1, y - 1); (\omega_2, 0)\} \end{aligned}$$

belong to \mathcal{K} in this case. There are two solutions to (1).

6. THE QUEUE WITH IMPATIENT CUSTOMERS

We now consider a queueing model with impatient customers. We use the same notation and assumptions as in section 5, except that customer C_n now requires to enter service before a given deadline, say at $T_n + D_n$. If not, the customer leaves the system at $T_n + D_n$ and is lost forever. We consider, that as soon as a customer entered the service booth, his service will proceed without interruption even though his patience elapses during his service. We assume that $\{D_n\}_{n \in \mathbb{Z}}$ is a sequence of marks of the arrival process, and that the generic r.v. D is non-negative. The system has a single server, operating in the order of arrivals (FIFO). Then (see [3, 5, 12]), the workload sequence $\{X_n\}_{n \in \mathbb{Z}}$ is stochastically recursive, driven by the mapping

$$\varphi_\omega(x) = [x + \sigma(\omega)\mathbf{1}_{\{x \leq D(\omega)\}} - \xi(\omega)]^+,$$

since each given customer is proposed a waiting time before entering service, that equals the workload just before his arrival time. Hence, the loss queue is a particular case of this model for $D = 0$ a.s., as easily understood. We aim once again to solve

$$(24) \quad Y \circ \theta = \varphi(Y), \text{ a.s..}$$

As can easily be checked, we have a.s. for all x ,

$$(25) \quad \chi_\omega(x) \leq \varphi_\omega(x) \leq \psi_\omega(x),$$

where

$$\begin{aligned} \chi(x) &= [x \vee (\sigma \wedge D) - \xi]^+, \\ \psi(x) &= [x \vee (\sigma + D) - \xi]^+ \end{aligned}$$

(see eq. (9) and (12) in [12]).

The random maps χ and ψ are a.s. continuous and non-decreasing, and the only proper solutions Y and Z to the recursions respectively driven by χ and ψ read

$$(26) \quad Y = \left[\max_{i \geq 1} \left((\sigma \wedge D) \circ \theta^{-i} - \sum_{j=1}^i \xi \circ \theta^{-j} \right) \right]^+,$$

$$(27) \quad Z = \left[\max_{i \geq 1} \left((\sigma + D) \circ \theta^{-i} - \sum_{j=1}^i \xi \circ \theta^{-j} \right) \right]^+.$$

Then, if $\mathbf{P}[Z = 0] > 0$, we are in the configuration of (iii) of Proposition 3, and a unique solution exists on the original probability space.

If $\mathbf{P}[Z = 0] = 0$, a construction based on tightness arguments is proposed in [12], that establishes the existence of a stationary workload on $\Omega \times \mathbb{R}_+$ provided that σ and ξ both take value in a set of the form

$$(28) \quad L_\alpha := \{n\alpha; n \in \mathbb{N}\}, \text{ where } \alpha \in \mathbb{R}_+.$$

In fact, Theorem 1 applies, and we can explicitly construct the extension in this case. Indeed, clearly $\varphi(x) \in L_\alpha$ a.s. for all $x \in L_\alpha$, so that the recursion, when initiated in L_α , remains in this set forever. We are thus in the case (iv) of Proposition 3 taking $L = L_\alpha$. More precisely, as in (17) and in view of (25), we have a.s. that for any $Y \leq x \leq Z$,

$$Y \circ \theta = \chi(Y) \leq \chi(x) \leq \varphi(x) \leq \psi(x) \leq \psi(Z) = Z \circ \theta.$$

Hence, the random set defined by

$$(29) \quad G = L_\alpha \cap [Y, Z] \text{ a.s.}$$

satisfies to (5). Moreover, Z is a.s. finite (see Lemma 6 below), thus G is a.s. of finite cardinal. So does H : (9) holds true and Theorem 1 applies.

Size of the extension. Let us investigate more precisely the form of the extension. First, denote

$$(30) \quad \underline{s} = \min \{n \in \mathbb{N}; \mathbf{P}[\sigma \leq n\alpha] > 0\},$$

$$(31) \quad \bar{s} = \inf \{n \in \mathbb{N}; \mathbf{P}[\sigma \leq n\alpha] = 1\},$$

$$(32) \quad \bar{d} = \inf \{n \in \mathbb{N}; \mathbf{P}[D \leq n\alpha] = 1\},$$

where \bar{s} and \bar{d} may be set to $+\infty$. Denote, a.s.,

$$(33) \quad \begin{aligned} A &= \left\{ i > 0; (D \circ \theta^{-i}) - \sum_{j=1}^i \xi \circ \theta^{-j} > 0 \right\} \text{ and } \tau^- = \sup A; \\ \tau^+ &= \min \left\{ i > 0; \sum_{j=0}^{i-1} \xi \circ \theta^j \geq D \right\}; \\ B &= \left\{ i > 0; (\sigma \circ \theta^{-i} + D \circ \theta^{-i}) - \sum_{j=1}^i \xi \circ \theta^{-j} > 0 \right\} \text{ and } \rho = \sup B. \end{aligned}$$

Each customer spends in the waiting line (resp. in the system: waiting line + service booth) a time at most equal to his/her patience time (resp. his/her service time plus his/her whole patience time). The set A (resp. B) thus contains all the absolute values of the indexes of the customers possibly in the waiting line (resp. in the total system) at time 0. Finally, τ^+ counts the number of arrivals customer 0 can see during his/her patience time.

Similarly to Lemma 4,

Lemma 6. *The r.v.'s ρ , τ^- and τ^+ are a.s. finite. In particular, there exist two integer p and t such that*

$$(34) \quad p = \min \{n > 0; \mathbf{P}[\rho \leq n] > 0\};$$

$$(35) \quad t = \min \{n > 0; \mathbf{P}[\tau^- \leq n] > 0\}.$$

As $H \subseteq G$ a.s., we have that

$$(36) \quad c \leq \text{Card} (L_\alpha \cap [Y; Z]) = \text{Card} \left(\mathbb{N} \cap \left[0, \frac{Z-Y}{\alpha} \right] \right) = \left\lceil \frac{Z-Y}{\alpha} \right\rceil \text{ a.s..}$$

Now set a.s.

$$\begin{aligned} i_0 &= \text{argmax} \left\{ \sigma \circ \theta^{-i} + D \circ \theta^{-i} - \sum_{j=1}^i \xi \circ \theta^{-j}; i \in \mathbb{N}^* \right\} \\ &= \text{argmax} \left\{ \sigma \circ \theta^{-i} + D \circ \theta^{-i} - \sum_{j=1}^i \xi \circ \theta^{-j}; i = 1, \dots, \rho \right\}. \end{aligned}$$

Then,

$$\begin{aligned} Z - Y &\leq \sigma \circ \theta^{-i_0} + D \circ \theta^{-i_0} - \sum_{j=1}^{i_0} \xi \circ \theta^{-j} \\ &\quad - \left((\sigma \circ \theta^{-i_0}) \wedge (D \circ \theta^{-i_0}) - \sum_{j=1}^{i_0} \xi \circ \theta^{-j} \right) \\ &= (\sigma \circ \theta^{-i_0}) \vee (D \circ \theta^{-i_0}) \text{ a.s.,} \end{aligned}$$

so that, with (36),

$$c \leq \left\lceil \frac{\max \left\{ (\sigma \circ \theta^{-i}) \vee (D \circ \theta^{-i}) ; i = 1, \dots, \rho \right\}}{\alpha} \right\rceil \text{ a.s..}$$

Therefore, on $\{\rho \leq p\}$,

$$(37) \quad c \leq \left\lceil \frac{\max_{i=1, \dots, p} \left\{ (\sigma \vee D) \circ \theta^{-i} \right\}}{\alpha} \right\rceil.$$

On another hand, the largest possible workload at time 0 is less than the sum of the service time of the customer in service (whose index has absolute value in B) and the service times requested by the customers in the waiting line at 0 (their indexes have absolute values in A). Therefore, a.s.,

$$\begin{aligned} (38) \quad H &\subseteq H^n \subseteq L_\alpha \cap \left(\bigcup_{i \in B} \left[0; \sigma \circ \theta^{-i} + \sum_{j=1}^{i-1} (\sigma \circ \theta^{-j}) \mathbf{1}_A\{j\} \right] \right) \\ &\subseteq L_\alpha \cap \left[0; \sigma \circ \theta^{-\rho} + \sum_{j=1}^{\rho-1} (\sigma \circ \theta^{-j}) \mathbf{1}_A\{j\} \right] \\ &\subseteq L_\alpha \cap \left[0; \sigma \circ \theta^{-\rho} + \sum_{j=1}^{\tau^-} (\sigma \circ \theta^{-j}) \mathbf{1}_A\{j\} \right] \\ &= L_\alpha \cap [0; M], \end{aligned}$$

where

$$(39) \quad M := \sigma \circ \theta^{-\rho} + \sum_{j=1}^{+\infty} (\sigma \circ \theta^{-j}) \mathbf{1}_A\{j\},$$

and where we use the fact that $\tau^- \leq \rho$, a.s.. This implies that a.s.

$$(40) \quad c \leq \left\lceil \frac{M}{\alpha} \right\rceil = \frac{M}{\alpha} + 1.$$

On the event $\{\rho \leq p\}$, we have $\tau^- \leq p$, thus $M \leq \sum_{j=1}^p (\sigma \circ \theta^{-j})$, and with (40),

$$(41) \quad c \leq \frac{\sum_{j=1}^p (\sigma \circ \theta^{-j})}{\alpha} + 1.$$

Now, on $\{\tau^- \leq t\}$,

$$M \leq \sigma \circ \theta^{-p} + \sum_{j=1}^t (\sigma \circ \theta^{-j}),$$

so with (40),

$$(42) \quad c \leq \frac{\max_{i=1,\dots,p} \sigma \circ \theta^{-i} + \sum_{j=1}^t (\sigma \circ \theta^{-j})}{\alpha} + 1.$$

The upper bounds (37), (41) and (42) hold with positive probability, hence they are true a.s. since c is deterministic. Therefore,

$$c \leq 1 + \frac{1}{\alpha} \cdot \min \left\{ \max_{i=1,\dots,p} (\sigma \circ \theta^{-i}) + \sum_{j=1}^t (\sigma \circ \theta^{-j}) ; \sum_{j=1}^p (\sigma \circ \theta^{-j}) ; \max_{i=1,\dots,p} \left((\sigma \circ \theta^{-i}) \vee (D \circ \theta^{-i}) \right) \right\} \text{ a.s..}$$

If we assume in particular that the service times are a.s. bounded, *i.e.* that \bar{s} defined by (31) is finite, we have that

$$c \leq \bar{s}((t+1) \wedge p) + 1,$$

and if additionally, the patience times are a.s. bounded (*i.e.* \bar{d} - defined by (32) - is finite), it follows that

$$c \leq \bar{s}((t+1) \wedge p) \wedge [\bar{s} \vee \bar{d}] + 1.$$

Now, remark that for all $j > 0$, a.s.

$$\tau^+ \circ \theta^{-j} > j \iff D \circ \theta^{-j} > \sum_{k=0}^{j-1} \xi \circ \theta^{k-j} \iff j \in A.$$

By the very definition of τ ,

$$\sum_{j=0}^{\tau^+-2} \xi \circ \theta^j < D,$$

so taking expectations, and then using θ -invariance we obtain

$$\begin{aligned} \mathbf{E}[D] &> \mathbf{E} \left[\sum_{j=1}^{\infty} \xi \circ \theta^{j-1} \mathbf{1}_{\tau^+ > j} \right] \\ &= \mathbf{E} \left[(\xi \circ \theta^{-1}) \sum_{j=1}^{\infty} \mathbf{1}_{\tau^+ \circ \theta^{-j} > j} \right] \\ (43) \quad &= \mathbf{E} [(\xi \circ \theta^{-1}) \text{Card } A]. \end{aligned}$$

Again, if we assume that \bar{s} is finite, it follows from (39) that

$$M \leq \bar{s}\alpha(1 + \text{Card } A) \text{ a.s.,}$$

so with (40),

$$c \leq \bar{s}(1 + \text{Card } A) + 1 \text{ a.s..}$$

Plugging this into (43), and using θ -invariance thus yields

$$(44) \quad c \leq \left\lceil \frac{\bar{s}(\mathbf{E}[D] + \mathbf{E}[\xi])}{\mathbf{E}[\xi]} \right\rceil.$$

All these results are collected in the following proposition.

Proposition 4. *A stationary workload exists on the original probability space whenever $\mathbf{P}[Z = 0] > 0$, where Z is defined by (27). If not, if both σ and ξ take value in L_α defined by (28), we have that*

$$(45) \quad c \leq 1 + \frac{1}{\alpha} \cdot \min \left\{ \max_{i=1, \dots, p} (\sigma \circ \theta^{-i}) + \sum_{j=1}^t (\sigma \circ \theta^{-j}) \right. \\ \left. ; \sum_{j=1}^p (\sigma \circ \theta^{-j}) ; \max_{i=1, \dots, p} \left((\sigma \vee D) \circ \theta^{-i} \right) \right\} \text{ a.s.,}$$

where p and t are defined respectively by (34) and (35).

If additionally, \bar{s} defined by (31) is finite,

$$c \leq \min \left\{ \bar{s}((t+1) \wedge p) + 1 ; \left\lceil \frac{\bar{s}(\mathbf{E}[D] + \mathbf{E}[\xi])}{\mathbf{E}[\xi]} \right\rceil \right\},$$

and if \bar{d} defined by (32) is as well finite,

$$c \leq \lfloor \bar{s} \vee \bar{d} \rfloor + 1.$$

Let us go through two examples to illustrate how our extension technique can be used to solve the stability problem of the queue with impatient customers. We work on the same dynamical system as in Examples 1 and 2.

Example 3

We first consider (a particular case of) Example 1, p. 303 in [12]. We set on $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$ the r.v.:

$$\begin{cases} \xi(\omega_1) &= \xi(\omega_2) = 1; \\ \sigma(\omega_1) &= 0.5, \sigma(\omega_2) = 1.5; \\ D(\omega_1) &= 1.51, D(\omega_2) = 2.01. \end{cases}$$

The workload is then valued in $0.5\mathbb{N}$, and we check that for this model, according to the definitions (26) and (27),

$$\begin{aligned} Z(\omega_1) &= \max \left\{ \sigma(\omega_2) + D(\omega_2) - \xi(\omega_2); \sigma(\omega_1) + D(\omega_1) - (\xi(\omega_2) + \xi(\omega_1)); \dots \right\} \\ &= 2.51; \end{aligned}$$

$$\begin{aligned} Z(\omega_2) &= \max \left\{ \sigma(\omega_1) + D(\omega_1) - \xi(\omega_1); \sigma(\omega_2) + D(\omega_2) - (\xi(\omega_1) + \xi(\omega_2)); \dots \right\} \\ &= 1.51; \end{aligned}$$

$$Y(\omega_1) = 0.5;$$

$$Y(\omega_2) = 0.$$

We thus start from the set $G = [Y, Z] \cap 0.5\mathbb{N}$, so

$$G_{\omega_1} = \{0.5, 1, \dots, 2.5\}, G_{\omega_2} = \{0, 0.5, \dots, 1.5\}.$$

It can be checked as well that

$$\begin{aligned} A_{\omega_1} &= \{1\} \text{ and } A_{\omega_2} = \{1, 2\}; \\ B_{\omega_1} &= \{1, 2, 3\} \text{ and } B_{\omega_2} = \{1, 2\}, \end{aligned}$$

so $t = 1$ and $p = 2$. The least upper-bound of c obtained according to Proposition 4 is given by

$$\lfloor \bar{s} \vee \bar{d} \rfloor + 1 = \lfloor 1.5 \vee 2.01 \rfloor + 1 = 3.$$

To explicit construct the extension, we now form the random sequence $\{H^n\}$ for the sample ω_1 :

$$\begin{aligned} H_{\omega_1}^1 &= \varphi_{\omega_2}(G_{\omega_2}) = \{0.5, 1, 1.5, 2\}; \\ H_{\omega_1}^2 &= \varphi_{\omega_2} \circ \varphi_{\omega_1}(G_{\omega_1}) = \varphi_{\omega_2}(\{0, 0.5, 1, 1.5\}) = \{0.5, 1, 1.5, 2\}; \\ H_{\omega_1}^3 &= \varphi_{\omega_2} \circ \varphi_{\omega_1} \circ \varphi_{\omega_2}(G_{\omega_2}) = \varphi_{\omega_2} \circ \varphi_{\omega_1}(\{0.5, 1, 1.5, 2\}) = \{0.5, 1, 1.5\}; \\ H_{\omega_1}^4 &= \varphi_{\omega_2} \circ \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ \varphi_{\omega_1}(G_{\omega_1}) = \varphi_{\omega_2} \circ \varphi_{\omega_1}(\{0.5, 1, 1.5, 2\}) = \{0.5, 1, 1.5\}; \\ H_{\omega_1}^5 &= \varphi_{\omega_2} \circ \varphi_{\omega_1}(\{0.5, 1, 1.5\}) = \{0.5, 1, 1.5\}; \\ H_{\omega_1}^6 &= \varphi_{\omega_2} \circ \varphi_{\omega_1}(\{0.5, 1, 1.5\}) = \{0.5, 1, 1.5\}; \\ &\vdots \end{aligned}$$

Therefore, $H_{\omega_1} = \{0.5, 1, 1.5\}$, and we obtain likewise that $H_{\omega_2} = \{0, 0.5, 1\}$. So $c = 3$ and there are at most three solutions on the original space.

In fact, it can be checked that the following events of \mathcal{F} :

$$\begin{aligned} \mathcal{J} &= \{(\omega_1, 0.5); (\omega_2, 0)\}, \\ \mathcal{J}' &= \{(\omega_1, 1); (\omega_2, 0.5)\}, \\ \mathcal{J}'' &= \{(\omega_1, 1.5); (\omega_2, 1)\} \end{aligned}$$

all belong to the set \mathcal{K} of Lemma 3. As a consequence, the extension is not ergodic, and the three corresponding r.v.'s X , X' and X'' are the only three stationary workloads on the original space.

Example 4

Now, set on the same probability space:

$$\begin{cases} \xi(\omega_1) = \xi(\omega_2) = 1; \\ \sigma(\omega_1) = 3, \sigma(\omega_2) = 2; \\ D(\omega_1) = 3.01, D(\omega_2) = 1.99. \end{cases}$$

The workload sequence is then valued in \mathbb{N} . As above, we first check that

$$\begin{aligned} Z(\omega_1) &= 4.01 \text{ and } Z(\omega_2) = 5.01, \\ Y(\omega_1) &= 1 \text{ and } Y(\omega_2) = 2, \\ A_{\omega_1} &= \{1, 2\} \text{ and } A_{\omega_2} = \{1, 3\}, \\ B_{\omega_1} &= \{1, 2, 3, 4, 6\} \text{ and } B_{\omega_2} = \{1, 2, 3, 5\}, \end{aligned}$$

so that $t = 2$ and $p = 5$. So the smallest upper-bound of c given by Proposition 4 is

$$\lfloor \bar{s} \vee \bar{d} \rfloor + 1 = 4.$$

Once again, we set $G = [Y, Z] \cap \mathbb{N}$ a.s., which amounts to

$$G_{\omega_1} = \{1, 2, 3, 4\} \text{ and } G_{\omega_2} = \{2, 3, 4, 5\}.$$

Then the computation yields

$$H_{\omega_1} = \{2, 3, 4\} \text{ and } H_{\omega_2} = \{3, 4, 5\},$$

therefore $c = 3$. It can be checked in that case, that the invariant sigma field of $\tilde{\mathcal{F}}$ is $\{\emptyset, \tilde{\Omega}\}$, so the extension is ergodic, but there is no stationary workload on the original probability space.

Independent case. We now address the case where the service times, patience times and inter-arrivals times form three independent i.i.d. sequences: the system is then denoted GI/GI/1/1+GI, and has been thoroughly studied *e.g.* by Baccelli *et al.* ([3, 5]).

First, assume that

$$(46) \quad \mathbf{P}[\sigma < \xi] > 0,$$

which implies, by independence, that $\mathbf{P}[\sigma \leq y - \varepsilon] > 0$ and $\mathbf{P}[\xi \geq y] > 0$ for some $y, \varepsilon > 0$.

The r.v. Z defined by (27) is finite, so there exists $n \in \mathbb{N}$ such that $\mathbf{P}[Z < n\varepsilon] > 0$. In particular, by θ -invariance,

$$\mathbf{P}[Z \circ \theta^{-n} < n\varepsilon] > 0.$$

Denote now for all $i \in \mathbb{N}^*$ and all $x \in \mathbb{R}+$ the events

$$\begin{aligned} \mathcal{E}_i^x &= \{\xi \circ \theta^{-j} \geq x; \forall j = 1, \dots, i\}; \\ \mathcal{F}_i^x &= \{\sigma \circ \theta^{-j} \leq x; \forall j = 1, \dots, i\}, \end{aligned}$$

fix a sample on the event

$$\mathcal{A}_n := \{Z \circ \theta^{-n} < n\varepsilon\} \cap \mathcal{E}_n^y \cap \mathcal{F}_n^{y-\varepsilon},$$

and assume that customer $-n$ finds upon arrival a workload w such that

$$w \in G_{\theta^{-n}\omega} = L_\alpha \cap [Y(\theta^{-n}\omega), Z(\theta^{-n}\omega)].$$

We are in the following alternative:

- (i) either for all $q = 1, \dots, n-1$, the workload $\Phi_{\theta^{-q}\omega}^{n-q}(w)$ upon the arrival of C_{-q} is positive, so the server never idles before the end of service of customer C_{-1} . In that case, the workload $\Phi_\omega^n(w)$ at 0 is less than the workload upon the arrival of customer $-n$ plus the work brought by the customers $C_{-n}, C_{-(n-1)}, \dots, C_{-1}$ minus the time elapsed, in other words

$$\Phi_\omega^n(w) \leq \left[w + \sum_{j=1}^n \sigma \circ \theta^{-j}(\omega) - \sum_{j=1}^n \xi \circ \theta^{-j}(\omega) \right]^+.$$

Hence, since $\omega \in \mathcal{A}_n$,

$$\begin{aligned} \Phi_\omega^n(w) &\leq [Z(\theta^{-n}\omega) + n(y - \varepsilon) - ny]^+ \\ &= 0. \end{aligned}$$

- (ii) or for some $q \in \{1, \dots, n-1\}$ (take the largest one), $\Phi_{\theta^{-q}\omega}^{n-q}(w) = 0$, the system is empty at the arrival of C_{-q} . Hence, since

$$\sigma \circ \theta^{-j}(\omega) < \xi \circ \theta^{-j}(\omega); j = 1, \dots, q,$$

each following customer is then immediately attended upon arrival and leaves the system before the next arrival, so $\Phi_{\theta^{-j}\omega}^{n-j}(w) = 0$ for all $j \in \{0, \dots, q\}$, and in particular $\Phi_\omega^n(w) = 0$.

Therefore, in any case and for any $w \in G_{\theta^{-n}\omega}$, $\Phi_{\omega}^n(w) = 0$. Thus, $H = \{0\}$, and in particular $c = 1$, on \mathcal{A}_n . It is now easy to check that $\mathbf{P}[\mathcal{A}_n] > 0$ since the events $\{Z \circ \theta^{-n} < n\varepsilon\}$, \mathcal{E}_n^y and $\mathcal{F}_n^{y-\varepsilon}$ are clearly independent and of positive probability. Thus, $c = 1$ a.s., the extension is ergodic and there exists a unique solution X on the original probability space, for which $\mathbf{P}[X = 0] > 0$. We hence capture again the stability result of Baccelli *et al.* (see [3, 5]).

Assume now that (46) does not hold. Notice that

$$\mathbf{P}\left[G_n^{s\alpha}\right] = (\mathbf{P}[\sigma \leq s\alpha])^n > 0,$$

where s is defined by (30). As above, it is then easily checked that the events $\{\rho \leq n\}$ and $G_n^{s\alpha}$ are independent, thus

$$\mathbf{P}\left[\{\rho \leq n\} \cap G_n^{s\alpha}\right] > 0.$$

On the latter event, M defined by (39) is such that $M \leq x(1 + \text{Card } A)$. Therefore the argument leading to (44) yields

$$\text{Card } H \leq \left\lceil \frac{s(\mathbf{E}[D] + \mathbf{E}[\xi])}{\mathbf{E}[\xi]} \right\rceil.$$

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